Signals & Systems
Solutions

S9.1 (a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad [sI-A] = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$ $\quad [sI-A]^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$

$$\Phi(t) = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

For $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $u(t) = F$, $t > 0$;

$$\vec{x}(t) = \int_0^t \Phi(t-\tau) B u(\tau)\, d\tau$$

$$\vec{x}(t) = \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} F\, d\tau = \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} F\, d\tau = \begin{bmatrix} Ft^2/2 \\ Ft \end{bmatrix}$$

state response is $\boxed{\vec{x}(t) = \begin{bmatrix} Ft^2/2 \\ Ft \end{bmatrix}}$

(b) Want $s = -1 \pm j \Rightarrow$ C.E. is $s^2 + 2s + 2 = 0$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[K_1 \; K_2] = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix}$$

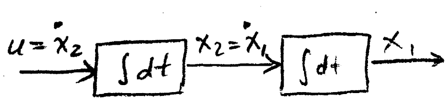
$$\det[sI - (A-BK)] = \det\begin{bmatrix} s & -1 \\ K_1 & s+K_2 \end{bmatrix} = s(s+K_2) + K_1 = 0$$

$$s^2 + sK_2 + K_1 = 0 \quad \Rightarrow \quad K_2 = 2, \; K_1 = 2$$

Controller is $\boxed{K = [2 \quad 2]}$

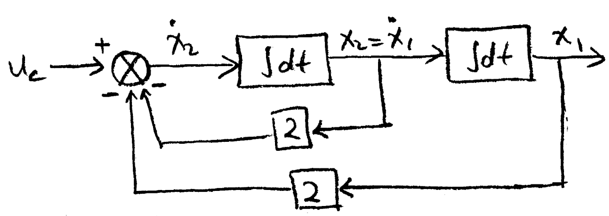
(c) uncontrolled

$\dot{x}_1 = x_2$, $\dot{x}_2 = u$



controlled $\quad A - BK = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$

$\dot{x}_1 = x_2$, $\dot{x}_2 = -2x_1 - 2x_2 + u_c$



S9.2 (a) $\det[sI - A] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & 30 \\ 0 & 0 & s+6 \end{bmatrix} = 0$

$$s\begin{vmatrix} s+1 & 30 \\ 0 & s+6 \end{vmatrix} + 1\begin{vmatrix} 0 & 30 \\ 0 & s+6 \end{vmatrix} = 0$$

$$s(s+1)(s+6) = 0. \Rightarrow \boxed{\text{Eigenvalues: } s=0,\ s=-1,\ s=-6}$$

(b) Desired C.E.

$$(s+7)(s+3-3j)(s+3+3j) = 0$$
$$(s+7)(s^2+6s-18) = 0$$
$$s^3 + 13s^2 + 60s + 126 = 0$$

$$[A-BK] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 30 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[K_1\ K_2\ K_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 30 \\ -K_1 & -K_2 & -6-K_3 \end{bmatrix}$$

$$\det[sI-(A-BK)] = \det\begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & -30 \\ K_1 & K_2 & s+6+K_3 \end{bmatrix} = 0$$

$$s[(s+1)(s+6+K_3)+30K_2] + 1[30K_1] = 0$$

$$s(s^2+7s+6+K_3+K_3 s+30K_2) + 30K_1 = 0$$

$$s^3 + (7+K_3)s^2 + (6+30K_2+K_3)s + 30K_1 = 0$$

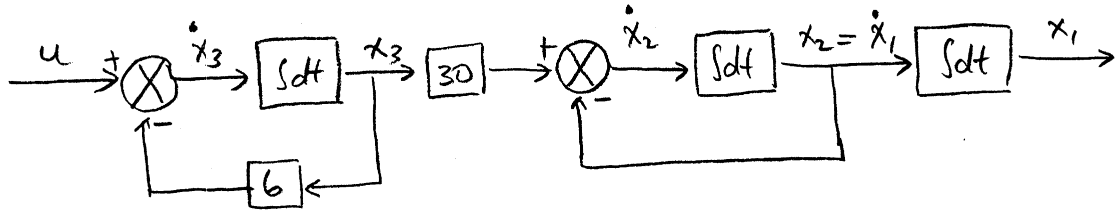
Comparing with $s^3 + 13s^2 + 60s + 126 = 0$, we see

$7+K_3 = 13 \Rightarrow K_3 = 6$; $30K_1 = 126 \Rightarrow K_1 = 4.2$; $6+30K_2+K_3 = 60 \Rightarrow K_2 = 1.6$

Controller is $\boxed{K = [4.2\ \ 1.6\ \ 6]}$

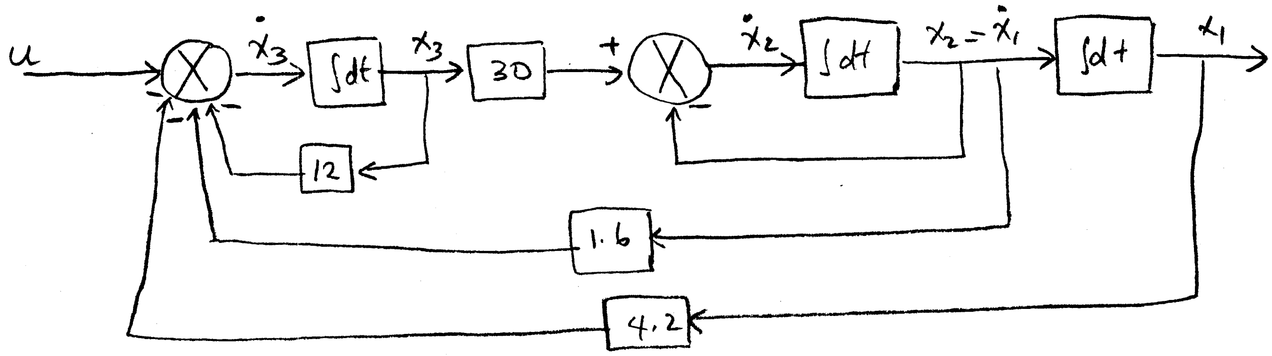
S9.2 (c)

uncontrolled: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + 30x_3, \quad \dot{x}_3 = -6x_3 + u$



controlled: $A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 30 \\ -4.2 & -1.6 & -12 \end{bmatrix}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + 30x_3$$
$$\dot{x}_3 = -4.2x_1 - 1.6x_2 - 12x_3 + u$$



Note the (negative) feedback we have added to stabilize the system.